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Lower and upper bounds for the anomalous diffusion exponent on Sierpinski carpets

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Abstract. The anomalous diffusion exponent d_w of random walks on a family of Sierpinski carpets are studied by analytical and numerical methods. We construct an effective bulk resistor and then establish the lower and upper bounds for d_w, where the lower bound turns out to be the same as that obtained with bond-moving renormalization. Numerical simulations on a family of Sierpinski carpets confirm our bounds, and show strong dependence of d_w on the lacunarity of the carpet.

Since Mandelbrot (1977, 1982) introduced the concept of fractals in physics, there have been many studies of various random walks on fractal lattices (Havlin and ben-Avraham 1987), but only a few studies on infinitely ramified fractals are found in the literature. As one of infinitely ramified fractals, Sierpinski carpets, whose geometrical properties are well established (Gefen *et al* 1983, Hao and Yang 1987), are studied for many instances; for examples, random walks (RW) (Gefen *et al* 1984a, Taguchi 1988a), self-avoiding walks (SAW) (Taguchi 1988b), directed Levy flight (Zhuang and Yao 1991), and directed self-avoiding walks (DSAW) (Kim *et al* 1992, 1993). The diffusion exponent d_w is defined as $\langle r^2(N)\rangle \sim N^{2/d_w}$, where $r(N)$ is the end-to-end distance of a random walker with N steps. The exact value of the exponent d_w is not known on Sierpinski carpets due to their infinite ramifications. Gefen *et al* (1984a) (more recently Taguchi 1988a) constructed resistor networks on carpets to calculate d_w approximately. The values of d_w obtained by these methods, however, do not agree very well with those of numerical simulations. In this paper we construct an effective bulk resistor to establish the lower and upper bounds for the exponent d_w on Sierpinski carpets. We also find an improved approximation of d_w, which agrees better with numerical simulations.

First we generate Sierpinski carpets whose generators are specified by the system size b and the hole size l (figure 1(a)). The fractal dimension of the carpet is $d_f = \log(b^2 - l^2)/\log b$. We then consider random walks on the carpet. Gefen *et al* (1984a) and Taguchi (1988a) constructed a resistor network by putting resistors on bonds between two subsquares in carpets. By using a bond-moving renormalization scheme (Migdal 1976, Kadanoff 1975), they obtained the resistance exponent $\tilde{\zeta}$, which is defined by $R(L) \sim L^{\tilde{\zeta}}$ for the system with linear size L. Then, using the fractal Einstein relation (Given and Mandelbrot 1988), $d_w = d_f + \tilde{\zeta}$, the diffusion exponent d_w is obtained.

Instead, we construct a bulk resistor in the following way. We consider the carpet as a metallic sheet with a hole and assign a unit resistance on each subsquare of carpets except the hole. We then get a bulk resistor with number of resistors $(b^2 - l^2)$ on the first generation of the carpet. In a bulk resistor with $b = 5$ and $l = 1$, for example, we consider the upper (and lower) 10 subsquares as a bulk resistor with resistance R_1, and middle four subsquares

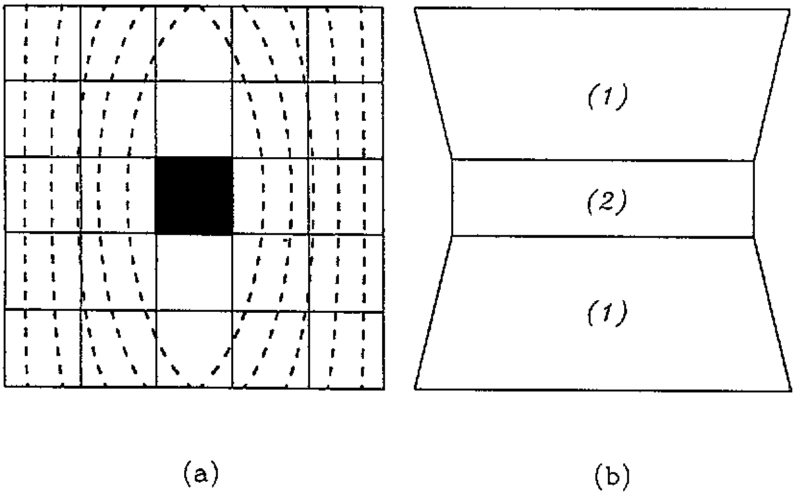

Figure 1. Construction of an effective bulk resistor. (*a*) Schematic electric field lines (broken curves) on the carpet. (*b*) Effective bulk resistor as a trapezoidal shape and rectangular shape, part (1) and part (2), respectively.

seperated by the hole as a bulk resistor with resistance R_2. These three resistors form a bulk resistor with total resistance of the first generation of carpet, $R_T(1) = 2R_1 + R_2$.

For homogeneous materials, a macroscopic resistance R is usually obtained by the macroscopic equation $R = \rho\frac{l}{A}$, which may be a good approximation for common conductors. Such macroscopic resistance depends only on the geometrical factors such as the length l and the cross-section area A. Because of the presence of the hole in the carpet, the electric field through the carpet is not uniform and it may not be so simple to find exact values of R_1 and R_2. If we neglect edge effects near holes and boundaries, however, we can use the macroscopic equation for R to calculate effective bulk resistances R_1^{eff} and R_2^{eff} approximately. Figure 1 exhibits the construction of an effective bulk resistor for R_1^{eff} and R_2^{eff} with $b = 5$ and $l = 1$ as an example. Since the electric field lines on the carpet are visualized as broken curves in figure 1(*a*), the upper and lower parts—part (1) in figure 1(*b*)—and the middle part—part (2) in figure 1(*b*)—may be approximately treated as a trapezoidal and a rectangular shape of bulk resistor, respectively. Using the macroscopic equation with $\rho = 1$, effective bulk resistances R_1^{eff} and R_2^{eff} for these shapes of effective bulk resistor can be easily calculated to give

$$R_1^{\mathrm{eff}} = \frac{b-l}{2l}\log\frac{b}{b-l} \qquad \text{and} \qquad R_2^{\mathrm{eff}} = \frac{l}{b-l}. \tag{1}$$

Thus total resistance of the first generation of carpet becomes

$$R_T^{\mathrm{eff}}(1) = \frac{b-l}{l}\log\frac{b}{b-l} + \frac{l}{b-l}. \tag{2}$$

For the second generation of carpet, the effective bulk resistance R_1' and R_2' can approximately be written in the same fashion to yield

$$R_1' = \frac{b-l}{2l}\log\frac{b}{b-l}R_T^{\mathrm{eff}}(1) \tag{3}$$

and

$$R_2' = \frac{l}{b-l} R_{\mathrm{T}}^{\mathrm{eff}}(1) . \tag{4}$$

Therefore the total effective resistance of the second generation becomes

$$R_{\mathrm{T}}^{\mathrm{eff}}(2) = 2R_1' + R_2' = \left[R_{\mathrm{T}}^{\mathrm{eff}}(1)\right]^2 . \tag{5}$$

By continuing these procedures to the nth generation of carpet, we have

$$R_{\mathrm{T}}^{\mathrm{eff}}(n) = \left[R_{\mathrm{T}}^{\mathrm{eff}}(1)\right]^n . \tag{6}$$

Since the linear size of the nth generation carpet is $L = b^n$, from (2) and (6), we get the resistance exponent $\tilde{\zeta}$:

$$\tilde{\zeta} = \log\left[\frac{b-l}{l}\log\frac{b}{b-l} + \frac{l}{b-l}\right](\log b)^{-1} . \tag{7}$$

According to the fractal Einstein relation ($d_{\mathrm{w}} = \tilde{\zeta} + d_f$), we thus obtain the effective exponent $d_{\mathrm{w}}^{\mathrm{eff}}$:

$$d_{\mathrm{w}}^{\mathrm{eff}} = \log\left[\left(\frac{b-l}{l}\log\frac{b}{b-l} + \frac{l}{b-l}\right)\left(b^2 - l^2\right)\right](\log b)^{-1} . \tag{8}$$

We can also establish upper and lower bounds for d_{w} easily in the following way. When one treats part (1) of the bulk resistor as a rectangle with the cross-sectional length $b-l$ (or b), one can obtain the maximum (or minimum) bulk resistance, $R_{1,\max} = (\frac{b-l}{2})(\frac{1}{b-l}) = \frac{1}{2}$ (or $R_{1,\min} = \frac{b-l}{2b}$). Following the above procedure, the upper and lower bounds for the effective exponent d_{w} are thus given by

$$d_{\mathrm{w}}^{\mathrm{upper}} = \log\left[\left(\frac{b-l}{b} + \frac{l}{b-l}\right)\left(b^2 - l^2\right)\right](\log b)^{-1} \tag{9}$$

$$d_{\mathrm{w}}^{\mathrm{lower}} = \log\left[\left(\frac{b}{b-l}\right)\left(b^2 - l^2\right)\right](\log b)^{-1} . \tag{10}$$

It would be worthy to note that our lower bound, equation (10), is the same as the result of an approximate Migdal–Kadanoff renormalization-group calculation obtained by Gefen *et al* (1984a) and corrected by Taguchi (1988a).

We have performed numerical simulations on a family of Sierpinski carpets to check the accuracy for our bounds of d_{w}. First, we have generated Sierpinski carpets using the generator (Zhuang and Yao 1991, Kim *et al* 1992). We then randomly choose a point and start a random walker. If a walker touches the boundary of carpets, we discard the walker and start a new walker at a newly chosen point. In order to get walks with a larger number of steps N, we select a starting point within a central part, which is seperated at least by $\sqrt{N}$ from the boundaries of the carpets. In this way we can generate RWs of $N = 10\,000$ steps without touching the boundaries of the carpets, ranging from the seventh generation for $(b, l) = (3, 1)$, the sixth generation for (4,2), the fifth generation for (5,1) and (5,3), and the fourth generation for (6,2), (7,1), (7,3), (8,2) and (8,3). The end-to-end distances averaged over 100 000 configurations are plotted versus $1/N$. The values of d_{w} are then obtained by least-square fitting. Numerical results on these carpets are shown in table 1 and compared with the lower and upper bounds. All numerical data are well located between the lower and upper bounds within numerical

Table 1. Comparison with numericsal data.

Carpet b, l	d_f	d_w^{lower}	d_w^{upper}	d_w^{eff}	d_w^{num}
3, 1	1.893	2.033	2.262	2.139	2.106 ± 0.016
4, 2	1.793	2.085	2.292	2.172	2.166 ± 0.016
5, 1	1.975	2.005	2.113	2.057	2.092 ± 0.017
5, 3	1.723	2.122	2.292	2.193	2.197 ± 0.014
6, 2	1.934	2.020	2.161	2.085	2.154 ± 0.017
7, 1	1.989	2.002	2.069	2.035	2.072 ± 0.016
7, 3	1.896	2.039	2.183	2.106	2.148 ± 0.014
8, 2	1.969	2.007	2.107	2.055	2.090 ± 0.015
8, 4	1.862	2.057	2.195	2.115	2.159 ± 0.015

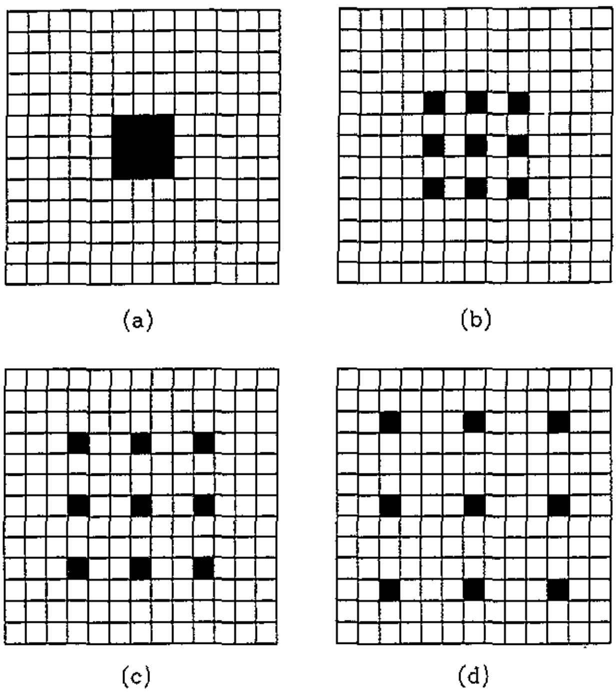

Figure 2. (13, 3) carpets with various lacunarities. (*a*) $L = 2.535$, (*b*) $L = 1.038$, (*c*) and (d) $L = 0.221$.

errors. From simulations, however, the results for the fourth generations of carpets seem to be close to the upper bounds.

We have also examined the dependence of d_w on the lacunarity which measures how the holes are distributed on the carpet without varying the value of b and l, i.e. d_f. Following Gefen *et al* (1984a), the lacunarity L is defined by

$$L \equiv \frac{1}{n} \sum_i (n_i - \langle n \rangle)^2 \qquad (11)$$

where $\langle n \rangle = \sum_i n_i / n$. Here n is the number of a $b \times b$ cell with $l \times l$ covering, and n_i is the number of non-empty subsquares for each ith covering. Figure 2 shows some examples

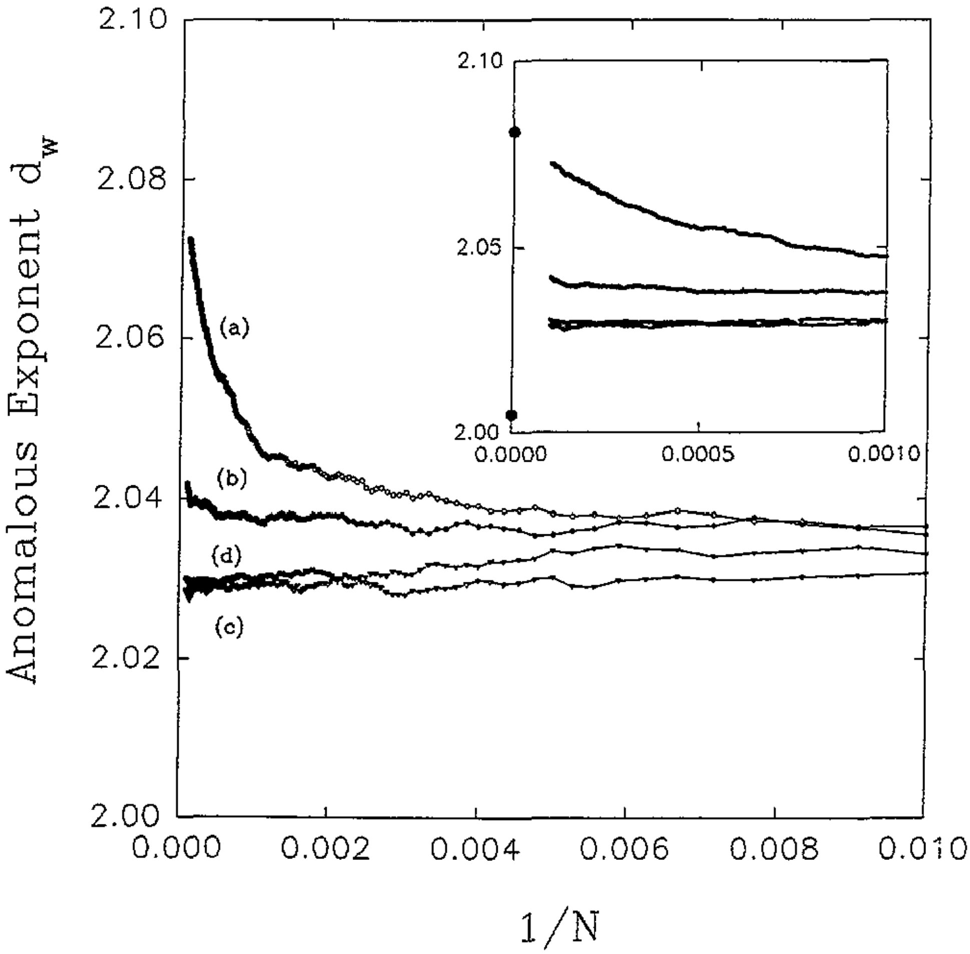

Figure 3. Dependence of d_w on the lacunarity for (13, 3) carpets. The upper and lower bounds for carpets are 2.081 and 2.005 (shown as full circles in inset), respectively. (*a*) $d_w = 2.078 \pm 0.016$, $L = 2.535$, (*b*) $d_w = 2.043 \pm 0.015$, $L = 1.038$, (*c*) $d_w = 2.028 \pm 0.016$, $L = 0.221$, and (*d*) $d_w = 2.028 \pm 0.17$, $L = 0.221$. More details for larger N are exhibited in the inset.

of (13, 3) carpets with various lacunarities, $L = 2.535$, 1.038 and 0.221. Our numerical results (shown in figure 3) are also well located within the the upper and lower bounds, but exhibit a strong dependence of d_w on the lacunarity. Note that two numerical results for same lacunaritiy ($L = 0.221$) are located at almost the same value of d_w. The carpets of different hole distributions seem to have different values of d_w, even if they have the same values of b and l. However, our approximate theory does not distinguish the systems' different lacunarity.

In summary, we have established the upper and lower bounds for the anomalous diffusion exponent on a family of Sierpinski carpets. Our lower bound turns out to be the same as the well known result by the renormalization group calculation. We also find an effective exponent for the comparison with numerical simulations. Our bounds are confirmed via numerical simulations on various carpets. The numerical simulations with the carpets of different lacunarities show that the value of d_w strongly depends on the lacunarity. Our approximate theory of d_w does not take into account differences in electric field due to lacunarities. It will be interesting to study how to incorporate the lacunarity effects into our theory.

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